Unitized Jordan algebras and dispersionless KdV equations

2001 J. Phys. A: Math. Gen. 34 2435

(http://iopscience.iop.org/0305-4470/34/11/332)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2435–2442 www.iop.org/Journals/ja PII: S0305-4470(01)14571-3

Unitized Jordan algebras and dispersionless KdV equations

O D McCarthy and I A B Strachan

Department of Mathematics, University of Hull, Kingston-upon-Hull HU6 7RX, UK

E-mail: o.d.mccarthy@maths.hull.ac.uk and i.a.strachan@maths.hull.ac.uk

Received 5 June 2000

Abstract
Multicomponent KdV systems are defined in terms of a set of structure constants and, as shown by Svinolupov, if these define a Jordan algebra the corresponding equations may be said to be integrable, at least in the sense of having higher-order symmetries, recursion operators and hierarchies of conservation laws. In this paper the dispersionless limits of these Jordan KdV equations are studied. Recursion laws for conserved densities are given under the assumption that the algebra possesses a unity element. Sufficient conditions are given for the unitized counterpart of a diagonalizable non-unital system to be diagonalizable. Hamiltonian structure is discussed within the context of D_N Jordan algebras and $\mathbb{CP}^N$ scattering problems.

PACS numbers: 0220, 0230, 0240, 0540

1. Introduction

The connection between Jordan algebras and multicomponent KdV was first obtained by Svinolupov [Sv]. He found that a necessary condition for the system

$$u^i_t = u^i_{xxx} + a^i_{jk} u^j u^k_x \qquad i, j, k = 1, \ldots, N \tag{1}$$

(where the fields u^i depend on x and t alone and the ${a_{jk}}^k$ are constants, symmetric in the lower indices) to possess higher symmetries and conservation laws was that the (commutative) algebra $\mathcal{J} = \langle e_i, i = 1, \ldots, N\rangle$ defined by the constants a^i_{jk} via

$$e_i \circ e_j = e^k_{ij} e_k$$

had to be Jordan; that is, for all $x, y \in \mathcal{J}$, $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$. The purpose of this paper is to study various properties of the dispersionless limits of this system, namely the hydrodynamic system

$$u^i_t = c^i_{jk} u^j u^k_x. \tag{2}$$

The ideas will be illustrated by means of the example

$$u^i_t = \left(u^i \sum_{j=1}^{N} u^j\right)_x \tag{3}$$

which is the dispersionless limit of the KdV system associated with a $\mathbb{CP}^N$-scattering problem. The structure constant are just

$$a^i_{jk} = \delta^i_j + \delta^i_k.$$

It is easy to show that this is a non-associative Jordan algebra which does not contain a unity element.

A basic result in the theory of Jordan algebras [Sc], but one which does not appear to have been utilized in the theory of integrable systems is the idea of unitizing an algebra. Suppose the Jordan algebra $\mathcal{J}$ does not contain an identity element (an example being the above algebra). Then one may adjoin an identity element e_0 to form a new algebra $\hat{\mathcal{J}} = \mathcal{J} \cup \langle e_0 \rangle$, with multiplication defined by

$$\begin{aligned}\hat{x} \circ \hat{y} &= (\lambda e_0 + x) \circ (\mu e_0 + y)\\ &= \lambda\mu e_0 + (\lambda y + \mu x + x \circ y).\end{aligned}$$

It is straightforward to show that $\hat{\mathcal{J}}$ is also Jordan:

$$\begin{aligned}(\hat{x}^2 \circ \hat{y}) \circ \hat{x} - \hat{x}^2 \circ (\hat{y} \circ \hat{x}) &= 2\lambda\{(x \circ y) \circ x - x \circ (y \circ x)\} + (x^2 \circ y) \circ x - x^2 \circ (y \circ x)\\ &= 0.\end{aligned}$$

Note that the result is false for generalizations of Jordan algebras where the commutativity condition is removed.

The unitized counterpart of the system (3) is the $(N + 1)$-component system

$$\begin{aligned}u^0_t &= u^0 u^0_x\\ u^i_t &= \left(u^i \sum_{j=0}^{N} u^j\right)_x \qquad i = 1, \ldots, N.\end{aligned} \tag{4}$$

The existence of a unity element enables a simple recursion scheme to be formulated for the conserved densities of the hydrodynamic system. This will be considered in section 2. Criteria for the diagonalizability of unitized systems are considered in section 3. In section 4 the Hamiltonian structure of the system (3) and its unitized counterpart (4) will be studied.

2. Conservation laws and the unity element

Owing to the commutativity of the algebra $\hat{\mathcal{J}}$ the hydrodynamic system automatically has N-conservation laws:

$$\left(u^i\right)_t = \left(\tfrac{1}{2} c^i_{jk} u^j u^k\right)_x.$$

Here a hierarchy of conserved densities $h^{(n)}$ will be constructed recursively [St1].

Any hydrodynamic conservation law

$$Q[\boldsymbol{u}]_t = \text{flux}[\boldsymbol{u}]_x$$

may be expanded, using (2), yielding

$$\frac{\partial\, \text{flux}}{\partial u^k} = a_{jk}{}^i u^j \frac{\partial Q}{\partial u^i}.$$

The integrability condition for this is

$$a_{jk}{}^{i}u^{j}\frac{\partial^{2}Q}{\partial u^{i}\partial u^{p}}=a_{jp}{}^{i}u^{j}\frac{\partial^{2}Q}{\partial u^{i}\partial u^{k}}. \tag{5}$$

By differentiating this with respect to u^0, the unity element, one finds that $\partial Q/\partial u^0$ satisfies the same equation, and hence is also conserved. These conserved densities are all homogeneous and may be labelled by their degree, so by Euler's theorem

$$u^{i}\frac{\partial h^{(n)}}{\partial u^{i}}=nh^{(n)}. \tag{6}$$

They may also be normalized so that

$$\frac{\partial h^{(n)}}{\partial u^{0}}=h^{(n-1)}. \tag{7}$$

The basic relation (5) may also be used to derive a recursion relation amongst the densities. Let $p=0$ in (5), so, on using the unity relation, equations (6) and (7) give

$$a_{jk}{}^{i}u^{j}\frac{\partial h^{(n-1)}}{\partial u^{i}}=(n-1)\frac{\partial h^{(n)}}{\partial u^{k}}. \tag{8}$$

Multiplying by u^k and using Euler's theorem (6) again yields

$$h^{(n)}=\frac{1}{n(n-1)}a_{jk}{}^{i}u^{j}u^{k}\frac{\partial h^{(n-1)}}{\partial u^{i}}. \tag{9}$$

Alternatively, from (8) and Euler's theorem one may derive

$$\left[\frac{\partial^{2}h^{(n)}}{\partial u^{i}\partial u^{j}}-c_{ij}{}^{k}\frac{\partial h^{(n-1)}}{\partial u^{k}}\right]u^{j}=0.$$

The term in square brackets will not, in general, be zero, but for Frobenius manifolds it does vanish, while in the example below it does not.

The above derivation holds, in part, for any commutative algebra with a unity. However, the derivation of (9) only uses a subset of the relation in (5); thus one must show that the $h^{(n)}$ constructed from $h^{(n-1)}$ via (9) satisfies all of the relations in (5). In general, there is an obstruction [St2], related to the failure of the algebra from being associative. For example, the unitized version of the dispersionless limit of Ito's system has none-vanishing obstructions. For simple systems, such as (3) and (4) one may bypass these general considerations and construct conservation laws directly.

Example. (The D_3-Jordan algebra). The only three-dimensional irreducible Jordan algebra is, up to isomorphisms, defined by the multiplication table

$$\begin{aligned} e_0\circ e_i &= +e_0\\ e_i\circ e_i &= -e_0\\ e_i\circ e_j &= 0 \qquad \text{for}\quad i\neq j\quad i\neq 0\quad j\neq 0. \end{aligned}$$

This gives rise to the hydrodynamic system

$$\begin{aligned} u_t &= -3(u^2-v^2-w^2)_x\\ v_t &= -6(uv)_x\\ w_t &= -6(uw)_x. \end{aligned} \tag{10}$$

The first few conserved densities are:

$$h^{(2)} = \frac{1}{2!}\left\{u^2 - v^2 - w^2\right\}$$
$$h^{(3)} = \frac{1}{3!}\left\{u^3 - 3u(v^2 + w^2)\right\}$$
$$h^{(4)} = \frac{1}{4!}\left\{u^4 - 6u^2(v^2 + w^2) + (v^2 + w^2)^2\right\}.$$

The general terms may easily by derived:

$$h^{(n)} = \frac{1}{n!}\sum_{r=0}^{[n/2]}(-1)^r \begin{pmatrix} n \\ 2r \end{pmatrix} (v^2 + w^2)^r\, u^{(n-2r)}.$$

These may be amalgamated into a generating function, the coefficients of λ in the power-series expansion being the conserved densities,

$$\mathcal{Q}(\lambda) = \mathrm{e}^{\lambda u}\cos\lambda\sqrt{v^2 + w^2}.$$

Similarly, there is a second family of conservation laws given by the generating function

$$\mathcal{Q}(\lambda) = \mathrm{e}^{\lambda u}\sin\lambda\sqrt{v^2 + w^2}.$$

These may be combined as

$$\mathcal{Q}(\lambda) = \mathrm{e}^{\lambda(u\pm \mathrm{i}\sqrt{v^2+w^2})}.$$

Example. For the simple systems (3) the function $f(\sum_{i=1}^{N} u^i)$ is a conserved density for any function f. For the unitized system (4) the function $f(u^0 + 2\sum_{i=1}^{N} u^i) + g(u^0)$ is conserved for any functions f and g. To satisfy the homogeneity and recursion formulae one may take

$$h^{(n)} = \frac{1}{n!}\left(u^0 + 2\sum_{i=1}^{N} u^i\right)^n + \frac{1}{n!}\left(u^0\right)^n.$$

3. Diagonalizable criteria for unitized systems

A necessary and sufficient condition for a hydrodynamic system

$$u^i{}_t = v^i{}_j(u)\, u^j{}_x$$

to be put into Riemann invariant form[1]

$$R^i{}_t = v^i[\boldsymbol{R}]R^i{}_x$$

is the vanishing of the Haantjes tensor [N, H]. This is defined in terms of the Nijenhuis tensor

$$N^i_{jk} = v^p_j\partial_p v^i_k - v^p_k\partial_p v^i_j - v^i_p(\partial_j v^p_k - \partial_k v^p_j)$$

by

$$T^i_{jk} = N^i_{pr}v^p_j v^r_k - N^p_{jr}v^i_p v^r_k - N^p_{rk}v^i_p v^r_j + N^p_{jk}v^i_r v^r_p.$$

[1] The summation convention is not used for diagonal coordinates.

If this vanishes then the hydrodynamic system is integrable using the generalized hodograph transform [T].

For any dispersionless Jordan KdV system the Nijenhuis tensor is linear:

$$N^i_{jk} = \Delta_{jrk}{}^i u^r$$

where

$$\Delta_{ijk}{}^s = a_{ij}{}^r a_{rk}{}^s - a_{jk}{}^r a_{ir}{}^s.$$

This is the associator of the algebra, the failure of the algebra from being associative:

$$e_i \circ (e_j \circ e_k) - (e_i \circ e_j) \circ e_k = \Delta^s_{ijk} e_s.$$

This shows that any dispersionless KdV equation arising from a commutative associative algebra is diagonalizable.

Suppose that a particular Jordan algebra without unity gives rise to a diagonalizable system, i.e. a vanishing Haantjes tensor. What can be said concerning the Nijenhuis and Haantjes tensor for the unitized system? It is easy to show that for any algebra with a unity element e_0 that $\Delta^s_{ijk} = 0$ if any one of the lower indices takes the value zero. For the unitized system it also follows that $\Delta^s_{ijk} = 0$ if the upper index is zero. It follows from this that if the original system has a vanishing Hanntjes tensor, the only possible non-vanishing terms for the Hanntjes tensor of the unitized system are $T^i_{0j}, i, j \neq 0$. Thus one has a reduced set of diagonalizability criteria.

If the Riemann invariants R^i for the non-unitized system are homogeneous functions of the variables $u^1, \ldots, u^N$ then it is straightforward to determine whether or not the unitized system is diagonalizable. Suppose that R^i is homogeneous of degree n_i; then by Euler's theorem,

$$\sum_{j=1}^N u^j \frac{\partial R^i}{\partial u^j} = n_i R^i. \tag{11}$$

If $\{v^i\},\ i = 1, \ldots, N$ are the speeds of the non-unitized system, then it is clear that the unitized system has speeds of $\{u^0, v^i + u^0\}$. It is also clear that u^0 is a Riemann invariant for the unitized system; set $R^0 = u^0$. Hence for the extended system, it is seen that

$$R^i{}_t = (v^i[\boldsymbol{R}] + R^0) R^i{}_x + n_i R^i R^0{}_x. \tag{12}$$

This simple relation shows that the Riemann invariants for the non-unitized system still play a role in the unitized system, reducing the number of entries in the hydrodynamic matrix from $(N^2 + N + 1)$ to $(2N + 1)$. In particular, if R^i is a homogenous function of degree zero, then R^i is also a Riemann invariant for the unitized system. Thus if $n_i = 0\ \forall i$, then the unitized system is diagonalizable and the Riemann invariants for the old system are also the Riemann invariants for the new system.

Now let $f^i(R^0, R^1, \ldots, R^N)$ be a smooth function. If this is a Riemann invariant for the unitized system, then it will satisfy

$$\frac{\partial f^i}{\partial t} = (v^i + R^0) \frac{\partial f^i}{\partial x}.$$

By equation (12), this is equivalent to the restriction

$$\left\{ \sum_{j=1}^N n_j R^j \frac{\partial f}{\partial R^j} - v^i \frac{\partial f^i}{\partial R^0} \right\} R^0{}_x + \sum_{j=1}^N (v^j - v^i) \frac{\partial f^i}{\partial R^j} R^j{}_x = 0.$$

For two distinct speeds, $v^i \neq v^j$, it follows from the above that $\partial f^i/\partial R^j = 0$. For simplicity, suppose that a hydrodynamic system is strictly hyperbolic, i.e. all speeds are distinct. (The more general case will be considered in [Mc1].) Then f^i is a function of R^0 and R^i only, and satisfies the equation

$$n_i R^i \frac{\partial f^i}{\partial R^i} = v^i \frac{\partial f^i}{\partial R^0}. \tag{13}$$

Differentiating both sides with respect to R^j, $i \neq j$, one finds that

$$\frac{\partial v^i}{\partial R^j} \cdot \frac{\partial f^i}{\partial R^0} = 0. \tag{14}$$

From equation (13), if $\partial f^i/\partial R^0 = 0$ then $n_i = 0$; that is, R^i is a function of degree zero. If $n_i \neq 0$, then it must be that $\partial v^i/\partial R^j = 0$, and so v^i is a function of R^i only. Hence the solution to equation (13) is

$$f^i(R^0, R^i) = \phi^i(r^i + n_i R^0) \tag{15}$$

where ϕ^i is any smooth function and

$$r^i = \int \frac{v^i(R^i)}{R^i}\, \mathrm{d}R^i.$$

It is thus seen that even if a non-unital system is diagonalizable, the corresponding unitized system is only diagonalizable if the Riemann invariants are functions of degree zero or the speeds v^i are functions of the Riemann invariant R^i only.

Example. Both systems (3) and (4) are diagonalizable. For these systems one may calculate the Riemann invariants directly, rather then relying on existence criteria.

For the non-unitized system (3), it is convenient to introduce the variable $S = \sum_{j=1}^{N} u^j$, allowing the system to be written as

$$u^i_t = (Su^i)_x. \tag{16}$$

It is immediately obvious that $S_t = (S^2)_x$. Now we introduce $N - 1$ quotients $R^\alpha = u^\alpha/u^1$, where $\alpha = 2, \ldots, N$. It is easily verified that the Riemann invariant form of equation (3) is given by

$$\begin{aligned} S_t &= 2SS_x \\ R^\alpha{}_t &= SR^\alpha{}_x. \end{aligned} \tag{17}$$

The Riemann invariants R^α of the non-unitized system are homogeneous functions of degree zero, and so are unchanged when a unity element is adjoined to the algebra. The Riemann invariant S is a homogeneous function of degree one; as the speed $2S$ is unique, equation (15) may be used without contradiction. Hence the transformed Riemann invariant is $R^1 = 2S + R^0$. It may be verified by direct computation that the Riemann invariant form of the unitized system (4) is

$$\begin{aligned} R^0{}_t &= R^0 R^0{}_x \\ R^1{}_t &= R^1 R^1{}_x \\ R^\alpha{}_t &= \tfrac{1}{2}(R^0 + R^1) R^\alpha{}_x \end{aligned} \tag{18}$$

where again $\alpha = 2, \ldots, N$. Note that the characteristic speeds of system (17) are S and $2S$, with multiplicity $N - 1$ and 1, respectively, whereas the speeds of the unitized system are u^0, $S + u^0$ and $2S + u^0$, with multiplicity 1, $N - 1$ and 1, respectively. Furthermore, note that when $N = 2$, equation (18) is just the Riemann invariant form of (10); in general, equation (18) is the Riemann invariant form of the dispersionless KdV equations for a D_{N+1} Jordan algebra.

4. Hamiltonian structure

A system of hydrodynamic type is said to be Hamiltonian if there exists a Hamiltonian $H = \int \mathrm{d}x\, h(\boldsymbol{u})$ and a Hamiltonian operator

$$\hat{A}^{ij} = g^{ij}(\boldsymbol{u})\frac{\mathrm{d}}{\mathrm{d}x} + b^{ij}{}_{k}(\boldsymbol{u})\, u^k_x \tag{19}$$

which defines a skew-symmetric Poisson bracket on functionals

$$\{F, G\} = \int \mathrm{d}x\, \frac{\delta F}{\delta u^i(x)} \hat{A}^{ij} \frac{\delta G}{\delta u^j(x)}$$

which satisfies the Jacobi identity and which generates the system

$$u^i_t = \{u^i(x), H\}.$$

Dubrovin and Novikov [DN] proved necessary and sufficient conditions for $\hat{A}^{ij}$ to be a Hamiltonian operator in the case when g^{ij} is not degenerate. These are:

(a) $\boldsymbol{g} = (g^{ij})^{-1}$ defines a (pseudo-)Riemannian metric;
(b) $b^{ij}{}_k = -g^{is}\Gamma^j{}_{sk}$, where $\Gamma^j{}_{sk}$ are the coefficients of the Levi-Civita connection;
(c) the Riemann curvature tensor of $\boldsymbol{g}$ is identically zero.

For diagonal systems, there is a simple formula for the metric coefficients [T], namely

$$\partial_i v_j = \tfrac{1}{2}(v_i - v_j)\partial_i \log g_{jj} \tag{20}$$

where $i \neq j$ and v_i, v_j are characteristic speeds. Note that the term $(v_i - v_j)$ normally appears as a denominator on the left-hand side. However, for systems (17) and (18) there are $N - 1$ repeated speeds, and so this would involve division by zero. For diagonal systems with repeated speeds $v_i = v_j$, formula (20) is valid, provided that $\partial_i v_j = 0$. This last condition is an example of what is sometimes referred to in the literature as *weak nonlinearity*.

Given a diagonalizable non-unital system whose unitized counterpart is also diagonalizable, the $(N + 1)$-dimensional metric may be computed as an extension of the original N-dimensional metric. Note, in particular, that the unitized system has speeds of $v^i + u^0$, $v^j + u^0$ and so the factor $(v^i - v^j)$ in Tsarev's formula (20) is unchanged.

Example. The Hamiltonian structure of the non-unitized system (17) is determined by the metric

$$\boldsymbol{g} = \phi(S)\, \mathrm{d}S^2 + S^2\psi(\boldsymbol{R})\, \mathrm{d}\boldsymbol{R}^2 \tag{21}$$

where $\psi(\boldsymbol{R})\, \mathrm{d}\boldsymbol{R}^2 = \psi(R^1, \ldots, R^{N-1})[(\mathrm{d}R^1)^2 + \cdots + (\mathrm{d}R^{N-1})^2]$, and $\phi(S)$, $\psi(\boldsymbol{R})$ are arbitrary functions of integration. In particular, when $N = 2$, $\phi(S) = 1$ and $\psi(\boldsymbol{R}) = 1$ the metric is

$$\boldsymbol{g} = \mathrm{d}S^2 + S^2\, \mathrm{d}R^2 \tag{22}$$

the metric for polar coordinates in $\mathbb{R}^2$. However, this is the *only* flat metric belonging to the class (21) of metrics. In general, the Hamiltonian structure of this system is non-local in the manner described by Ferapontov [F]. Such structures have been calculated explicitly in [Mc2].

The Hamiltonian structure of the unitized system (18) is given by the metric

$$\hat{\boldsymbol{g}} = \alpha(R^0)(\mathrm{d}R^0)^2 + \tfrac{1}{4}\phi(\tfrac{1}{2}R^1)(\mathrm{d}R^1)^2 + \tfrac{1}{4}(R^1 - R^0)^2\psi(\boldsymbol{R})\, \mathrm{d}\boldsymbol{R}^2 \tag{23}$$

where α is some arbitrary function. Note that metric (23) reduces to (21) if the coordinate reduction $R^0 = 0$ is imposed, since $R^1 = 2S + R^0$. Analysis of metric (23) reveals that for

$N = 2$, the (three-dimensional) metric is flat provided α and ϕ are constant. For $N \geqslant 3$ the ($(N+1)$-dimensional metric) is flat if and only if $\alpha = \frac{1}{4}$, $\phi = -4$ and $\psi(\boldsymbol{R}) = 1$. Hence the unitized system has a single local Hamiltonian system for all N, whereas the non-unitized system has a local structure when $N = 2$ only. There is also sufficient functional freedom in these metrics to give a bi-Hamiltonian structure.

5. Comments

The idea of unitizing a dispersionless Jordan KdV system has been illustrated in this paper by means of an example based on the dispersionless limit of a $\mathbb{CP}^N$-valued scattering problem. The idea, however, is more general and may be applied to other systems, notably those with scattering problems associated with Hermitian symmetric spaces [FK]. More generally still, the idea may be applied directly to the dispersive system (1). The properties of such systems still need to be fully investigated.

In terms of their Hamiltonian structures, unitizing the system (3) corresponds to moving from a curved (sub)manifold to its flat ambient space. Thus the curved submanifold is of codimension, or embedding class, one for all valued of N. A classical result due to Cartan states that an N-dimensional manifold may be embedded in a flat ambient space of dimension (at most) $N(N+1)/2$, and so has embedding class (at most) $N(N-1)/2$. The codimension is also reflected in the length of the non-local tail in the corresponding Hamiltonian structure [F].

A question which one may therefore ask at this stage is whether or not dispersionless unital Jordan KdV equations always possess a flat metric. For all well known simple Jordan algebras with unity, there is a non-degenerate inner product on the algebra which may be used to define a flat metric. However, there are many other Jordan algebras to be considered and a study of these will contribute to a more complete understanding of integrable structures.

Acknowledgments

The first author would like to thank the University of Hull for a research Scholarship and the London Mathematical Society for supporting his attendance at the MMRD meeting.

References

[DN] Dubrovin B A and Novikov S P 1983 The Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyabov–Whitham averaging method *Sov. Math. Dokl.* **27** 665–9

[F] Ferapontov E V 1995 Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications *Topics in Topology and Mathematical Physics (Am. Math. Soc. Transl. Ser. 2 vol 170)* ed S P Novikov

[FK] Fordy A P and Kulish P P 1983 Nonlinear Schödinger equations and simple Lie algebras *Commun. Math. Phys.* **89** 427–33

[H] Haantjes J 1955 On X_m forming sets of eigenvectors *Nederl. Akad. Wet.* A **58** 158–62

[Mc1] McCarthy O D 2000 Dispersionless integrable systems of KdV type *PhD Thesis* University of Hull

[Mc2] McCarthy O D 1999 A simple family of non-local Poisson brackets *J. Nonlin. Math. Phys.* (*Proc. NEEDS 99*) **8** suppl, at press

[N] Nijenhuis A 1951 X_{m-1} forming sets of eigenvectors *Nederl. Akad. Wet.* A **54** 200–12

[Sc] Schafer R D 1966 *An Introduction to Nonassociative Algebras* (New York: Academic)

[St1] Strachan I A B 2000 Jordan manifolds and dispersionless KdV equations *J. Nonlin. Math. Phys.* **7** 495–510

[St2] Strachan I A B 2000 Deformations of dispersionless KdV hierarchies *Preprint* nlin.SI/0010016

[Sv] Svinolupov S I 1991 Jordan algebras and generalised Korteweg–de Vries equations *Teor. Math. Phys.* **87** 611–20

[T] Tsarev S P 1985 On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type *Sov. Math. Dokl.* **31** 488–91